The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

cemc.uwaterloo.ca

2018 Canadian Team Mathematics Contest

Individual Problems

IMPORTANT NOTES:

- Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) previously stored information such as formulas, programs, notes, etc., (iv) a computer algebra system, (v) dynamic geometry software.
- Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

PROBLEMS:

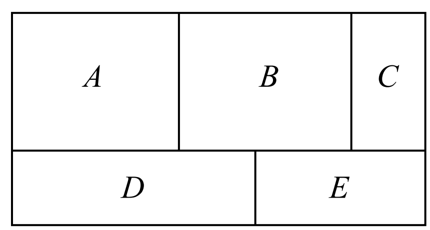
1. The point with coordinates $(a, 0)$ is on the line with equation $y = x + 8$. What is the value of a?

2. If

$$x = \left(1 - \tfrac{1}{12}\right)\left(1 - \tfrac{1}{11}\right)\left(1 - \tfrac{1}{10}\right)\left(1 - \tfrac{1}{9}\right)\left(1 - \tfrac{1}{8}\right)\left(1 - \tfrac{1}{7}\right)\left(1 - \tfrac{1}{6}\right)\left(1 - \tfrac{1}{5}\right)\left(1 - \tfrac{1}{4}\right)\left(1 - \tfrac{1}{3}\right)\left(1 - \tfrac{1}{2}\right)$$

what is the value of x?

3. In the diagram, a large rectangle is divided into five smaller rectangles which are labelled A, B, C, D, E. In how many ways can exactly two of these five rectangles be shaded so that the shaded rectangles are not touching?



4. The length of the diagonal of a square is 10. What is the area of this square?

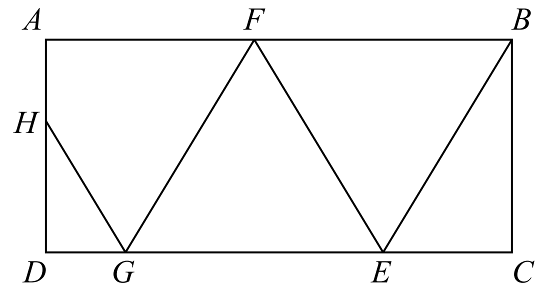
5. A three-digit positive integer n has digits abc. (That is, a is the hundreds digit of n, b is the tens digit of n, and c is the ones (units) digit of n.) Determine the largest possible value of n for which
   - a is divisible by 2,
   - the two-digit integer ab (that, a is the tens digit and b is the ones (units) digit) is divisible by 3 but is not divisible by 6, and
   - n is divisible by 5 but is not divisible by 7.

6. Determine all pairs of real numbers (x, y) for which $(4x^2 - y^2)^2 + (7x + 3y - 39)^2 = 0$.

7. An arithmetic sequence has a common difference, d, that is a positive integer and is greater than 1. The sequence includes the three terms 3, 468 and 2018. What is the sum of all of the possible values of d?

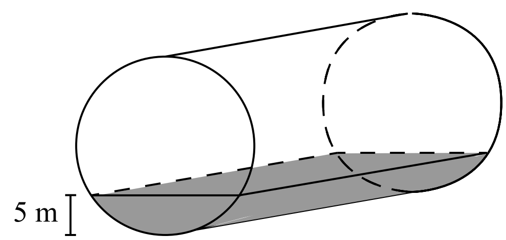
   (An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant, called the common difference. For example, $3, 5, 7, 9$ are the first four terms of an arithmetic sequence with common difference 2.)

8. Rectangular room $ABCD$ has mirrors on walls AB and DC. A laser is placed at B. It is aimed at E and the beam reflects off of the mirrors at E, F and G, arriving at H. The laws of physics tell us that $\angle BEC = \angle FEG$ and $\angle BFE = \angle AFG$ and $\angle FGE = \angle HGD$. If $AB = 18$ m, $BC = 10$ m and $HD = 6$ m, what is the total length of the path $BEFGH$ travelled by the laser beam?



9. A box contains R red balls, B blue balls, and no other balls. One ball is removed and set aside, and then a second ball is removed. On each draw, each ball in the box is equally likely to be removed. The probability that both of these balls are red is $\frac{2}{7}$. The probability that exactly one of these balls is red is $\frac{1}{2}$. Determine the pair (R, B).

10. A cylindrical tank has radius 10 m and length 30 m. The tank is lying on its side on a flat surface and is filled with water to a depth of 5 m. Water is added to the tank and the depth of the water increases from 5 m to $10 + 5\sqrt{2}$ m. If the volume of water added to the tank, in m^3, can be written as $a\pi + b + c\sqrt{p}$ for some integers a, b, c and prime number p, determine the quadruple (a, b, c, p).



2018 Canadian Team Mathematics Contest

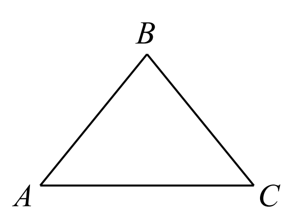
RELAY QUESTIONS

*The acronym **TNYWR** (\the number you will receive")
indicates the answer from the previous student.

0 (a). Evaluate $\dfrac{9+3\times 3}{3}$.

0 (b). Let t be TNYWR.
What is the area of a triangle with base $2t$ and height $3t+2$?

0 (c). Let t be TNYWR.
In the diagram, $\triangle ABC$ is isosceles with $AB = BC$. If $\angle ABC = t^\circ$, what is the measure of $\angle BAC$, in degrees?



1 (a). The integers 390 and 9450 have three common positive divisors that are prime numbers. What is the sum of these prime numbers?

---

1 (b). Let t be TNYWR.

If $n = \dfrac{(4t^2 - 10t - 2) - 3(t^2 - t + 3) + (t^2 + 5t - 1)}{(t + 7) + (t - 13)}$, what is the value of n?

---

1 (c). Let t be TNYWR.

Azmi has two fair dice, each with six sides.

The sides of one of the dice are labelled $1, 2, 3, 4, 5, 6$.

The sides of the other die are labelled $t - 10, t, t + 10, t + 20, t + 30, t + 40$.

When these two dice are rolled, there are 36 different possible values for the sum of the numbers on the top faces. What is the average of these 36 possible sums?

2 (a). The expression $2(x-3)^2 - 12$ can be re-written as $ax^2 + bx + c$ for some numbers a, b, c. What is the value of $10a - b - 4c$?

---

2 (b). Let t be TNYWR.

The line ℓ passes through the points $(-4, t)$ and (k, k) for some real number k.

The line ℓ is perpendicular to the line passing through the points $(11, -7)$ and $(15, 5)$.

What is the value of k?

---

2 (c). Let t be TNYWR.

In a magic square, the sum of the numbers in each column, the sum of the numbers in each row, and the sum of the numbers on each diagonal are all the same. In the magic square shown, what is the value of N?

	$3t-2$	$4t-6$	
$4t-1$	$2t+12$	$t+16$	$3t+1$
N	$4t-2$		$t+15$
			$4t-5$

3 (a). The line with equation $y = 5x + a$ passes through the point (a, a^2). If $a \neq 0$, what is the value of a?

---

3 (b). Let t be TNYWR.

The CEMC Compasses basketball team scored exactly $10t$ points in each of 4 games and scored exactly 20 points in each of g games. Over this set of games, they scored an average of 28 points per game. What is the value of g?

---

3 (c). Let t be TNYWR.

The pair $(x, y) = (a, b)$ is a solution of the system of equations

$$x^2 + 4y = t^2$$
$$x^2 - y^2 = 4$$

If $b > 0$, what is the value of b?

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Team Problems

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PROBLEMS:

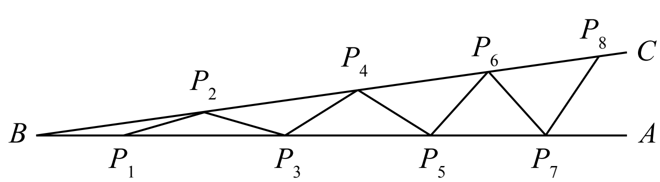
1. What is the value of $\sqrt{1+2+3+4+5+6+7+8}$?

2. A bucket that is $\frac{2}{3}$ full contains 9 L of maple syrup. What is the capacity of the bucket, in litres?

3. The sum of four consecutive odd integers is 200. What is the largest of these four integers?

4. It takes 18 doodads to make 5 widgets. It takes 11 widgets to make 4 thingamabobs. How many doodads does it take to make 80 thingamabobs?

5. In the diagram, points P_1, P_3, P_5, P_7 are on BA and points P_2, P_4, P_6, P_8 are on BC such that $BP_1 = P_1P_2 = P_2P_3 = P_3P_4 = P_4P_5 = P_5P_6 = P_6P_7 = P_7P_8$. If $\angle ABC = 5^\circ$, what is the measure of $\angle AP_7P_8$?



6. A *polygonal pyramid* is a three-dimensional solid. Its base is a regular polygon. Each of the vertices of the polygonal base is connected to a single point, called the *apex*. The sum of the number of edges and the number of vertices of a particular polygonal pyramid is 1915. How many faces does this pyramid have?

7. There are four ways to evaluate the expression "$\pm 2 \pm 5$":

$$2+5=7 \qquad 2-5=-3 \qquad -2+5=3 \qquad -2-5=-7$$

There are eight ways to evaluate the expression "$\pm 2^{11} \pm 2^5 \pm 2$". When these eight values are listed in decreasing order, what is the third value in the list?

8. For how many positive integers n is the sum

$$(-n)^3 + (-n+1)^3 + \cdots + (n-2)^3 + (n-1)^3 + n^3 + (n+1)^3$$

less than 3129?

9. For real numbers a and b, we define $a \nabla b = ab - ba^2$. For example, $5 \nabla 4 = 5(4) - 4(5^2) = -80$. Determine the sum of the values of x for which $(2 \nabla x) - 8 = x \nabla 6$.

10. Birgit has a list of four numbers. Luciano adds these numbers together, three at a time, and gets the sums 415, 442, 396, and 325. What is the sum of Birgit's four numbers?

11. On Monday, Krikor left his house at 8:00 a.m., drove at a constant speed, and arrived at work at 8:30 a.m. On Tuesday, he left his house at 8:05 a.m., drove at a constant speed, and arrived at work at 8:30 a.m. By what percentage did he increase his speed from Monday to Tuesday?

12. What is the value of $\pi \log_{2018} \sqrt{2} + \sqrt{2} \log_{2018} \pi + \pi \log_{2018} \left(\frac{1}{\sqrt{2}}\right) + \sqrt{2} \log_{2018} \left(\frac{1}{\pi}\right)$?

13. An Eilitnip number is a three-digit positive integer with the properties that, for some integer k with $0 \le k \le 7$:

    - its digits are k, $k+1$ and $k+2$ in some order, and
    - it is divisible by $k+3$.

    Determine the number of Eilitnip numbers.

14. An arithmetic sequence has 2036 terms labelled $t_1, t_2, t_3, \ldots, t_{2035}, t_{2036}$. Its 2018th term is $t_{2018} = 100$. Determine the value of $t_{2000} + 5t_{2015} + 5t_{2021} + t_{2036}$.

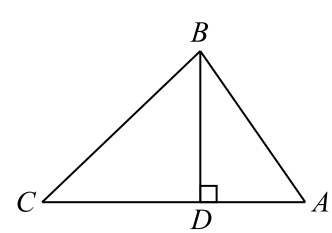
    (An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant, called the common difference. For example, $3, 5, 7, 9$ are the first four terms of an arithmetic sequence with common difference 2.)

15. A square wall has side length n metres. Guillaume paints n non-overlapping circular targets on the wall, each with radius 1 metre. Mathilde is going to throw a dart at the wall. Her aim is good enough to hit the wall at a single point, but poor enough that the dart will hit a random point on the wall. What is the largest possible value of n so that the probability that Mathilde's dart hits a target is at least $\frac{1}{2}$?

16. Determine the largest positive integer n for which 7^n is a divisor of the integer $\dfrac{200!}{90!30!}$.

    (Note: If n is a positive integer, the symbol $n!$ (read "n factorial") is used to represent the product of the integers from 1 to n. That is, $n! = n(n-1)(n-2)\cdots(3)(2)(1)$. For example, $5! = 5(4)(3)(2)(1)$ or $5! = 120$.)

17. In the diagram, D is on side AC of $\triangle ABC$ so that BD is perpendicular to AC. Also, $\angle BAC = 60^\circ$ and $\angle BCA = 45^\circ$. If the area of $\triangle ABC$ is $72 + 72\sqrt{3}$, what is the length of BD?



18. BBAAABA and AAAAAAA are examples of "words" that are seven letters long where each letter is either A or B. How many seven-letter words in which each letter is either A or B do not contain three or more A's in a row?

19. Determine all real numbers x for which $x^{3/5} - 4 = 32 - x^{2/5}$.

20. In $\triangle ABC$, $BC = AC - 1$ and $AC = AB - 1$. If $\cos(\angle BAC) = \frac{3}{5}$, determine the perimeter of $\triangle ABC$.

21. The function f has the following properties:

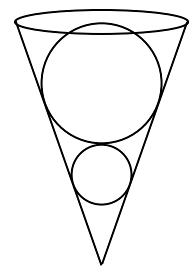
    (i) its domain is all real numbers,

    (ii) it is an odd function (that is, $f(-x) = -f(x)$ for every real number x), and

    (iii) $f(2x-3) - 2f(3x-10) + f(x-3) = 28 - 6x$ for every real number x.

    Determine the value of $f(4)$.

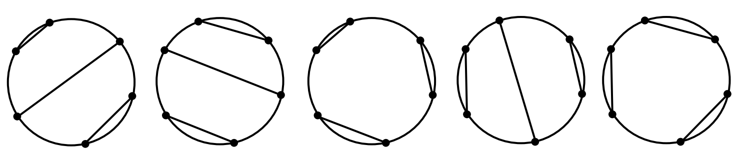
22. A right circular cone contains two spheres, as shown. The radius of the larger sphere is 2 times the radius of the smaller sphere. Each sphere is tangent to the other sphere and to the lateral surface of the cone. The larger sphere is tangent to the cone's circular base. Determine the fraction of the cone's volume that is *not* occupied by the two spheres.



23. Let a be a fixed real number. Define $M(t)$ to be the maximum value of $-x^2 + 2ax + a$ over all real numbers x with $x \leq t$. Determine a polynomial expression in terms of a that is equal to $M(a-1) + M(a+2)$ for every real number a.

24. The graphs $y = 2\cos 3x + 1$ and $y = -\cos 2x$ intersect at many points. Two of these points, P and Q, have x-coordinates between $\frac{17\pi}{4}$ and $\frac{21\pi}{4}$. The line through P and Q intersects the x-axis at B and the y-axis at A. If O is the origin, what is the area of $\triangle BOA$?

25. There are 16 distinct points on a circle. Determine the number of different ways to draw 8 non-intersecting line segments connecting pairs of points so that each of the 16 points is connected to exactly one other point. (For example, when the number of points is 6, there are 5 different ways to connect them, as shown.)



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2018
Canadian Team Mathematics Contest

April 2018

Solutions

Individual Problems

1. Since $(a, 0)$ is on the line with equation $y = x + 8$, then $0 = a + 8$ or $a = -8$.

   Answer: -8

2. Simplifying,

   $$\begin{aligned} x &= \left(1 - \tfrac{1}{12}\right)\left(1 - \tfrac{1}{11}\right)\left(1 - \tfrac{1}{10}\right)\left(1 - \tfrac{1}{9}\right)\left(1 - \tfrac{1}{8}\right)\left(1 - \tfrac{1}{7}\right)\left(1 - \tfrac{1}{6}\right)\left(1 - \tfrac{1}{5}\right)\left(1 - \tfrac{1}{4}\right)\left(1 - \tfrac{1}{3}\right)\left(1 - \tfrac{1}{2}\right) \\ &= \left(\tfrac{11}{12}\right)\left(\tfrac{10}{11}\right)\left(\tfrac{9}{10}\right)\left(\tfrac{8}{9}\right)\left(\tfrac{7}{8}\right)\left(\tfrac{6}{7}\right)\left(\tfrac{5}{6}\right)\left(\tfrac{4}{5}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{1}{2}\right) \\ &= \tfrac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} \\ &= \tfrac{1}{12} \qquad \text{(dividing out common factors)} \end{aligned}$$

   Answer: $\frac{1}{12}$

3. The following pairs of rectangles are not touching: AC, AE, CD.
   There are 3 such pairs.

   Answer: 3

4. Let the side length of the square be s.
   Since the diagonal of length 10 is the hypotenuse of a right-angled triangle with two sides of the square as legs, then $s^2 + s^2 = 10^2$ or $2s^2 = 100$, which gives $s^2 = 50$.
   Since the area of the square equals s^2, then the area is 50.

   Answer: 50

5. To make n as large as possible, we make each of the digits a, b, c as large as possible, starting with a.
   Since a is divisible by 2, its largest possible value is $a = 8$, so we try $a = 8$.
   Consider the two digit integer $8b$.
   This integer is a multiple of 3 exactly when $b = 1, 4, 7$.
   We note that 84 is divisible by 6, but 81 and 87 are not.
   To make n as large possible, we try $b = 7$, which makes $n = 87c$.
   For $n = 87c$ to be divisible by 5, it must be the case that $c = 0$ or $c = 5$.
   But $875 = 7 \cdot 125$ so 875 is divisible by 7.
   Therefore, for n to be divisible by 5 and not by 7, we choose $c = 0$.
   Thus, the largest integer n that satisfies the given conditions is $n = 870$.

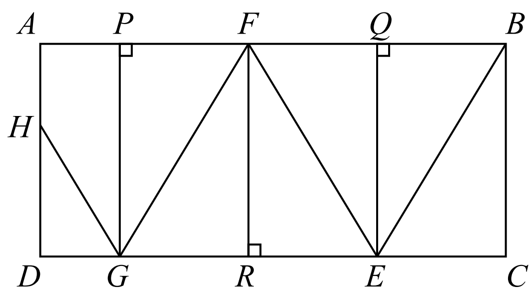
   Answer: 870

6. First, we note that $a^2 \geq 0$ for every real number a.
   Therefore, $(4x^2 - y^2)^2 \geq 0$ and $(7x + 3y - 39)^2 \geq 0$.
   Since $(4x^2 - y^2)^2 + (7x + 3y - 39)^2 = 0$, then it must be the case that $(4x^2 - y^2)^2 = 0$ and $(7x + 3y - 39)^2 = 0$.
   This means that $4x^2 - y^2 = 0$ and $7x + 3y - 39 = 0$.
   The equation $4x^2 - y^2 = 0$ is equivalent to $4x^2 = y^2$ and to $y = \pm 2x$.
   If $y = 2x$, then the equation $7x + 3y - 39 = 0$ becomes $7x + 6x = 39$ or $x = 3$.
   This gives $y = 2x = 6$.
   If $y = -2x$, then the equation $7x + 3y - 39 = 0$ becomes $7x - 6x = 39$ or $x = 39$.
   This gives $y = -2x = -78$.
   Therefore, the solutions to the original equation are $(x, y) = (3, 6), (39, -78)$.

   Answer: $(x, y) = (3, 6), (39, -78)$

7. In an arithmetic sequence with common difference d, the difference between any two terms must be divisible by d. This is because to get from any term in the sequence to any term later in the sequence, we add the common difference d some number of times.
In the given sequence, this means that $468 - 3 = 465$ is a multiple of d and $2018 - 468 = 1550$ is a multiple of d.
Thus, we want to determine the possible positive common divisors of 465 and 1550.
We note that $465 = 5 \cdot 93 = 3 \cdot 5 \cdot 31$ and that $1550 = 50 \cdot 31 = 2 \cdot 5^2 \cdot 31$.
Therefore, the positive common divisors of 465 and 1550 are $1, 5, 31, 155$. (These come from finding the common prime divisors.)
Since $d > 1$, the possible values of d are $5, 31, 155$. The sum of these is $5 + 31 + 155 = 191$.
ANSWER: 191

8. Let points P and Q be on AB so that GP and EQ are perpendicular to AB.
Let point R be on DC so that FR is perpendicular to DC.



Note that each of $\triangle BCE$, $\triangle EQB$, $\triangle EQF$, $\triangle FRE$, $\triangle FRG$, and $\triangle FPG$ is right-angled and has height equal to BC, which has length 10 m.
Since $\angle BEC = \angle FEG$, then the angles of $\triangle BCE$ and $\triangle FRE$ are equal. Since their heights are also equal, these triangles are congruent.
Since $FREQ$ and $BCEQ$ are rectangles (each has four right angles) and each is split by its diagonal into two congruent triangles, then $\triangle EQF$ and $\triangle EQB$ are also congruent to $\triangle BCE$.
Similarly, $\triangle FPG$ and $\triangle FRG$ are congruent to these triangles as well.
Let $CE = x$ m. Then $ER = RG = EC = x$ m.
Since $\triangle HDG$ is right-angled at D and $\angle HGD = \angle FGE$, then $\triangle HGD$ is similar to $\triangle FRG$.
Since $HD : FR = 6 : 10$, then $DG : RG = 6 : 10$ and so $DG = \frac{6}{10} RG = \frac{3}{5} x$ m.
Since $AB = 18$ m and $ABCD$ is a rectangle, then $DC = 18$ m.
But $DC = DG + RG + ER + EC = \frac{3}{5}x + 3x = \frac{18}{5}x$ m.
Thus, $\frac{18}{5}x = 18$ and so $x = 5$.
Since $x = 5$, then by the Pythagorean Theorem in $\triangle BCE$,

$$BE = \sqrt{BC^2 + EC^2} = \sqrt{(10\text{ m})^2 + (5\text{ m})^2} = \sqrt{125\text{ m}^2} = 5\sqrt{5}\text{ m}$$

Now $FG = EF = BE = 5\sqrt{5}$ m and $GH = \frac{6}{10} FG = \frac{3}{5}(5\sqrt{5}\text{ m}) = 3\sqrt{5}$ m.
Therefore, the length of the path $BEFGH$ is $3(5\sqrt{5}\text{ m}) + (3\sqrt{5}\text{ m})$ or $18\sqrt{5}$ m.
ANSWER: $18\sqrt{5}$ m

9. The box contains $R+B$ balls when the first ball is drawn and $R+B-1$ balls when the second ball is drawn.
Therefore, there are $(R+B)(R+B-1)$ ways in which two balls can be drawn.
If two red balls are drawn, there are R balls that can be drawn first and $R-1$ balls that can be drawn second, and so there are $R(R-1)$ ways of doing this.
Since the probability of drawing two red balls is $\frac{2}{7}$, then $\frac{R(R-1)}{(R+B)(R+B-1)} = \frac{2}{7}$.
If only one of the balls is red, then the balls drawn are either red then blue or blue then red.
There are RB ways in the first case and BR ways in the second case, since there are R red balls and B blue balls in the box.
Since the probability of drawing exactly one red ball is $\frac{1}{2}$, then $\frac{2RB}{(R+B)(R+B-1)} = \frac{1}{2}$.
Dividing the first equation by the second, we obtain successively

$$\begin{aligned}
\frac{R(R-1)}{(R+B)(R+B-1)} \cdot \frac{(R+B)(R+B-1)}{2RB} &= \frac{2}{7}\cdot\frac{2}{1} \\
\frac{R-1}{2B} &= \frac{4}{7} \\
R &= \frac{8}{7}B+1
\end{aligned}$$

Substituting into the second equation, we obtain successively

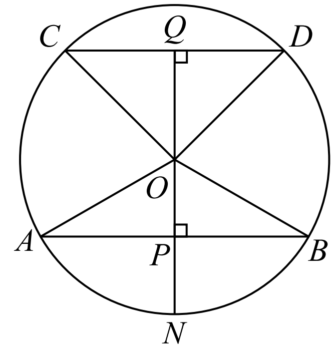
$$\begin{aligned}
\frac{2\left(\frac{8}{7}B+1\right)B}{\left(\frac{8}{7}B+1+B\right)\left(\frac{8}{7}B+1+B-1\right)} &= \frac{1}{2} \\
\frac{2\left(\frac{8}{7}B+1\right)B}{\left(\frac{15}{7}B+1\right)\left(\frac{15}{7}B\right)} &= \frac{1}{2} \\
\frac{2(8B+7)}{15\left(\frac{15}{7}B+1\right)} &= \frac{1}{2} \qquad \text{(since } B \neq 0\text{)} \\
32B+28 &= \tfrac{225}{7}B+15 \\
13 &= \tfrac{1}{7}B \\
B &= 91
\end{aligned}$$

Since $R = \frac{8}{7}B+1$, then $R = 105$ and so $(R,B) = (105,91)$.
(We can verify that the given probabilities are correct with these starting numbers of red and blue balls.)

Answer: $(R,B) = (105,91)$

10. Consider the front face of the tank, which is a circle of radius 10 m.
Suppose that when the water has depth 5 m, its surface is along horizontal line AB.
Suppose that when the water has depth $(10+5\sqrt{2})$ m, its surface is along horizontal line CD.
Let the area of the circle between the chords AB and CD be x m^2.
Since the tank is a cylinder which is lying on a flat surface, the volume of water added can be viewed as an irregular prism with base of area x m^2 and length 30 m.
Thus, the volume of water equals $30x$ m^3. Therefore, we need to calculate the value of x.

Let O be the centre of the circle, N be the point where the circle touches the ground, P the midpoint of AB, and Q the midpoint of CD.
Join O to A, B, C, and D. Also, join O to N and to Q.
Since Q is the midpoint of chord CD and O is the centre of the circle, then OQ is perpendicular to CD.
Since P is the midpoint of AB, then ON passes through P and is perpendicular to AB.
Since AB and CD are parallel and OP and OQ are perpendicular to these chords, then $QOPN$ is a straight line segment.



Since the radius of the circle is 10 m, then $OC = OD = OA = ON = OB = 10$ m.
Since AB is 5 m above the ground, then $NP = 5$ m.
Since $ON = 10$ m, then $OP = ON - NP = 5$ m.
Since CD is $(10 + 5\sqrt{2})$ m above the ground and $ON = 10$ m, then $QO = 5\sqrt{2}$ m.
Since $\triangle AOP$ is right-angled at P and $OP : OA = 1 : 2$, then $\triangle AOP$ is a 30°-60°-90° triangle with $\angle AOP = 60°$. Also, $AP = \sqrt{3}OP = 5\sqrt{3}$ m.
Since $\triangle CQO$ is right-angled at Q and $OC : OQ = 10 : 5\sqrt{2} = 2 : \sqrt{2} = \sqrt{2} : 1$, then $\triangle CQO$ is a 45°-45°-90° triangle with $\angle COQ = 45°$. Also, $CQ = OQ = 5\sqrt{2}$ m.

We are now ready to calculate the value of x.
The area between AB and CD is equal to the area of the circle minus the combined areas of the region under AB and the region above CD.
The area of the region under AB equals the area of sector AOB minus the area of $\triangle AOB$.
The area of the region above CD equals the area of sector COD minus the area of $\triangle COD$.
Since $\angle AOP = 60°$, then $\angle AOB = 2\angle AOP = 120°$.
Since $\angle COQ = 45°$, then $\angle COD = 2\angle COQ = 90°$.
Since the complete central angle of the circle is 360°, then sector AOB is $\frac{120°}{360°} = \frac{1}{3}$ of the whole circle and sector COD is $\frac{90°}{360°} = \frac{1}{4}$ of the whole circle.
Since the area of the entire circle is $\pi(10\text{ m})^2 = 100\pi\text{ m}^2$, then the area of sector AOB is $\frac{100}{3}\pi\text{ m}^2$ and the area of sector COD is $25\pi\text{ m}^2$.
Since P and Q are the midpoints of AB and CD, respectively, then $AB = 2AP = 10\sqrt{3}$ m and $CD = 2CQ = 10\sqrt{2}$ m.
Thus, the area of $\triangle AOB$ is $\frac{1}{2} \cdot AB \cdot OP = \frac{1}{2}(10\sqrt{3}\text{ m})(5\text{ m}) = 25\sqrt{3}\text{ m}^2$.
Also, the area of $\triangle COD$ is $\frac{1}{2} \cdot CD \cdot OQ = \frac{1}{2}(10\sqrt{2}\text{ m})(5\sqrt{2}\text{ m}) = 50\text{ m}^2$.
This means that the area of the region below AB is $\left(\frac{100}{3}\pi - 25\sqrt{3}\right)\text{ m}^2$ and the area of the region above CD is $(25\pi - 50)\text{ m}^2$.
Finally, this means that the volume of water added, in m^3, is

$$\begin{aligned} 30x &= 30\left(100\pi - \left(\frac{100}{3}\pi - 25\sqrt{3}\right) - (25\pi - 50)\right) \\ &= 3000\pi - 1000\pi + 750\sqrt{3} - 750\pi + 1500 \\ &= 1250\pi + 1500 + 750\sqrt{3} \end{aligned}$$

Therefore, $a\pi + b + c\sqrt{p} = 1250\pi + 1500 + 750\sqrt{3}$ and so $(a, b, c, p) = (1250, 1500, 750, 3)$.

ANSWER: $(1250, 1500, 750, 3)$

Team Problems

1. Evaluating,
$$\sqrt{1+2+3+4+5+6+7+8} = \sqrt{36} = 6$$
ANSWER: 6

2. Since the bucket is $\frac{2}{3}$ full and contains 9 L of maple syrup, then if it were $\frac{1}{3}$ full, it would contain $\frac{1}{2}(9\text{ L}) = 4.5$ L.
Therefore, the capacity of the full bucket is $3 \cdot (4.5\text{ L}) = 13.5$ L.
ANSWER: 13.5 L

3. Since the four integers are consecutive odd integers, then they differ by 2.
Let the four integers be $x-6, x-4, x-2, x$.
Since the sum of these integers is 200, then $(x-6)+(x-4)+(x-2)+x = 200$.
Simplifying and solving, we obtain $4x - 12 = 200$ and $4x = 212$ and $x = 53$.
Therefore, the largest of the four integers is 53.
ANSWER: 53

4. Since $80 = 20 \cdot 4$, then to make $80 = 20 \cdot 4$ thingamabobs, it takes $20 \cdot 11 = 220$ widgets.
Since $220 = 44 \cdot 5$, then to make $220 = 44 \cdot 5$ widgets, it takes $44 \cdot 18 = 792$ doodads.
Therefore, to make 80 thingamabobs, it takes 792 doodads.
ANSWER: 792

5. Since $BP_1 = P_1P_2 = P_2P_3 = P_3P_4 = P_4P_5 = P_5P_6 = P_6P_7 = P_7P_8$, then each of $\triangle BP_1P_2$, $\triangle P_1P_2P_3$, $\triangle P_2P_3P_4$, $\triangle P_3P_4P_5$, $\triangle P_4P_5P_6$, $\triangle P_5P_6P_7$, and $\triangle P_6P_7P_8$ is isosceles.
Since $\angle ABC = 5^\circ$, then $\angle BP_2P_1 = \angle ABC = 5^\circ$.
Next, $\angle P_2P_1P_3$ is an exterior angle for $\triangle BP_1P_2$.
Thus, $\angle P_2P_1P_3 = \angle P_1BP_2 + \angle P_1P_2B = 10^\circ$.
(To see this in another way, $\angle BP_1P_2 = 180^\circ - \angle P_1BP_2 - \angle P_1P_2B$ and
$$\angle P_2P_1P_3 = 180^\circ - \angle BP_1P_2 = 180^\circ - (180^\circ - \angle P_1BP_2 - \angle P_1P_2B) = \angle P_1BP_2 + \angle P_1P_2B$$
The first of these equations comes from the sum of the angles in the triangle and the second from supplementary angles.)
Continuing in this way,
$$\begin{aligned}
\angle P_2P_3P_1 &= \angle P_2P_1P_3 = 10^\circ \\
\angle P_3P_2P_4 &= \angle P_2P_3P_1 + \angle P_2BP_3 = 15^\circ \\
\angle P_3P_4P_2 &= \angle P_3P_2P_4 = 15^\circ \\
\angle P_4P_3P_5 &= \angle P_3P_4P_2 + \angle P_3BP_4 = 20^\circ \\
\angle P_4P_5P_3 &= \angle P_4P_3P_5 = 20^\circ \\
\angle P_5P_4P_6 &= \angle P_4P_5P_3 + \angle P_4BP_5 = 25^\circ \\
\angle P_5P_6P_4 &= \angle P_5P_4P_6 = 25^\circ \\
\angle P_6P_5P_7 &= \angle P_5P_6P_4 + \angle P_5BP_6 = 30^\circ \\
\angle P_6P_7P_5 &= \angle P_6P_5P_7 = 30^\circ \\
\angle P_7P_6P_8 &= \angle P_6P_7P_5 + \angle P_6BP_7 = 35^\circ \\
\angle P_7P_8P_6 &= \angle P_7P_6P_8 = 35^\circ \\
\angle AP_7P_8 &= \angle P_7P_8P_6 + \angle P_7BP_8 = 40^\circ
\end{aligned}$$
ANSWER: 40°

6. Suppose that the base of the pyramid has n sides.
The base will also have n vertices. Since the pyramid has one extra vertex (the apex), then it has $n+1$ vertices in total.
The pyramid has $n+1$ faces: the base plus n triangular faces formed by each edge of the base and the apex.
The pyramid has $2n$ edges: the n sides that form the base plus one edge joining each of the n vertices of the base to the apex.
From the given information, $2n+(n+1)=1915$.
Thus, $3n=1914$ and so $n=638$.
Since the pyramid has $n+1$ faces, then it has 639 faces.

ANSWER: 639

7. Since $2^{11}=2048$ and $2^5=32$, the eight values are

$$\begin{array}{cccc} 2^{11}+2^5+2=2082 & 2^{11}+2^5-2=2078 & 2^{11}-2^5+2=2018 & 2^{11}-2^5-2=2014 \\ -2^{11}+2^5+2=-2014 & -2^{11}+2^5-2=-2018 & -2^{11}-2^5+2=-2078 & -2^{11}-2^5-2=-2082 \end{array}$$

The third largest value is $2^{11}-2^5+2=2018$.

ANSWER: $2^{11}-2^5+2=2018$

8. For every real number a, $(-a)^3=-a^3$ and so $(-a)^3+a^3=0$.
Therefore,
$$(-n)^3+(-n+1)^3+\cdots+(n-2)^3+(n-1)^3+n^3+(n+1)^3$$
which equals
$$((-n)^3+n^3)+((-n+1)^3+(n-1)^3)+\cdots+((-1)^3+1^3)+0^3+(n+1)^3$$
is equal to $(n+1)^3$.
Since $14^3=2744$ and $15^3=3375$ and n is an integer, then $(n+1)^3<3129$ exactly when $n+1\leq 14$.
There are 13 positive integers n that satisfy this condition.

ANSWER: 13

9. Using the given definition, the following equations are equivalent:

$$\begin{aligned} (2\nabla x)-8 &= x\nabla 6 \\ (2x-4x)-8 &= 6x-6x^2 \\ 6x^2-8x-8 &= 0 \\ 3x^2-4x-4 &= 0 \end{aligned}$$

The sum of the values of x that satisfy the original equation equals the sum of the roots of this quadratic equation.
This sum equals $-\frac{-4}{3}$ or $\frac{4}{3}$.
(We could calculate the roots and add these, or use the fact that the sum of the roots of the quadratic equation $ax^2+bx+c=0$ is $-\dfrac{b}{a}$.)

ANSWER: $\frac{4}{3}$

10. Suppose Birgit's four numbers are a, b, c, d.
This means that the totals $a+b+c$, $a+b+d$, $a+c+d$, and $b+c+d$ are equal to 415, 442, 396, and 325, in some order.
If we add these totals together, we obtain

$$\begin{aligned}(a+b+c)+(a+b+d)+(a+c+d)+(b+c+d) &= 415+442+396+325 \\ 3a+3b+3c+3d &= 1578 \\ a+b+c+d &= 526\end{aligned}$$

since the order of addition does not matter.
Therefore, the sum of Luciano's numbers is 526.

ANSWER: 526

11. Let v_1 km/h be Krikor's constant speed on Monday.
Let v_2 km/h be Krikor's constant speed on Tuesday.
On Monday, Krikor drives for 30 minutes, which is $\frac{1}{2}$ hour.
Therefore, on Monday, Krikor drives $\frac{1}{2}v_1$ km.
On Tuesday, Krikor drives for 25 minutes, which is $\frac{5}{12}$ hour.
Therefore, on Tuesday, Krikor drives $\frac{5}{12}v_2$ km.
Since Krikor drives the same distance on both days, then $\frac{1}{2}v_1 = \frac{5}{12}v_2$ and so $v_2 = \frac{12}{5} \cdot \frac{1}{2}v_1 = \frac{6}{5}v_1$.
Since $v_2 = \frac{6}{5}v_1 = \frac{120}{100}v_1$, then v_2 is 20% larger than v_1.
That is, Krikor drives 20% faster on Tuesday than on Monday.

ANSWER: 20%

12. Using logarithm laws,

$$\begin{aligned}&\pi \log_{2018}\sqrt{2} + \sqrt{2}\log_{2018}\pi + \pi\log_{2018}\left(\frac{1}{\sqrt{2}}\right) + \sqrt{2}\log_{2018}\left(\frac{1}{\pi}\right) \\ &\quad = \log_{2018}\left(\sqrt{2}^{\pi}\right) + \log_{2018}\left(\pi^{\sqrt{2}}\right) + \log_{2018}\left(\frac{1}{\sqrt{2}^{\pi}}\right) + \log_{2018}\left(\frac{1}{\pi^{\sqrt{2}}}\right) \\ &\quad = \log_{2018}\left(\frac{\sqrt{2}^{\pi}\cdot\pi^{\sqrt{2}}}{\sqrt{2}^{\pi}\cdot\pi^{\sqrt{2}}}\right) \\ &\quad = \log_{2018}(1) \\ &\quad = 0\end{aligned}$$

ANSWER: 0

13. We make a table that lists, for each possible value of k, the digits, the possible three-digit integers made by these digits, and $k+3$:

k	$k, k+1, k+2$	Possible integers	$k+3$
0	0, 1, 2	102, 120, 201, 210	3
1	1, 2, 3	123, 132, 213, 231, 312, 321	4
2	2, 3, 4	234, 243, 324, 342, 423, 432	5
3	3, 4, 5	345, 354, 435, 453, 534, 543	6
4	4, 5, 6	456, 465, 546, 564, 645, 654	7
5	5, 6, 7	567, 576, 657, 675, 756, 765	8
6	6, 7, 8	678, 687, 768, 786, 867, 876	9
7	7, 8, 9	789, 798, 879, 897, 978, 987	10

When $k = 0$, the sum of the digits of each three-digit integer is 3, so each is divisible by 3.
When $k = 1$, only two of the three-digit integers are even: 132 and 312. Each is divisible by 4.
When $k = 2$, none of the three-digit integers end in 0 or 5 so none is divisible by 5.
When $k = 3$, only two of the three-digit integers are even: 354 and 534. Each is divisible by 6.
When $k = 4$, the integer 546 is divisible by 7. The rest are not. (One way to check this is by dividing each by 7.)
When $k = 5$, only two of the three-digit integers are even: 576 and 756. Only 576 is divisible by 8.
When $k = 6$, the sum of the digits of each of the three-digit integers is 21, which is not divisible by 9, so none of the integers is divisible by 9.
When $k = 7$, none of the three-digit integers end in 0 so none is divisible by 10.
In total, there are $4+2+2+1+1 = 10$ three-digit integers that satisfy the required conditions.

ANSWER: 10

14. Suppose that d is the common difference in this arithmetic sequence.
Since $t_{2018} = 100$ and t_{2021} is 3 terms further along in the sequence, then $t_{2021} = 100 + 3d$.
Similarly, $t_{2036} = 100 + 18d$ since it is 18 terms further along.
Since $t_{2018} = 100$ and t_{2015} is 3 terms back in the sequence, then $t_{2015} = 100 - 3d$.
Similarly, $t_{2000} = 100 - 18d$ since it is 18 terms back.
Therefore,

$$\begin{aligned} t_{2000} + 5t_{2015} + 5t_{2021} + t_{2036} &= (100 - 18d) + 5(100 - 3d) + 5(100 + 3d) + (100 + 18d) \\ &= 1200 - 18d - 15d + 15d + 18d \\ &= 1200 \end{aligned}$$

ANSWER: 1200

15. The area of the square wall with side length n metres is n^2 square metres.
The combined area of n circles each with radius 1 metre is $n \cdot \pi \cdot 1^2$ square metres or $n\pi$ square metres.
Given that Mathilde hits the wall at a random point, the probability that she hits a target is the ratio of the combined areas of the targets to the area of the wall, or $\dfrac{n\pi \text{ m}^2}{n^2 \text{ m}^2}$, which equals $\dfrac{\pi}{n}$.
For $\dfrac{\pi}{n} > \dfrac{1}{2}$, it must be the case that $n < 2\pi \approx 6.28$.
The largest value of n for which this is true is $n = 6$.

ANSWER: $n = 6$

16. First, we count the number of factors of 7 included in 200!.
Every multiple of 7 includes least 1 factor of 7.
The product 200! includes 28 multiples of 7 (since $28 \times 7 = 196$).
Counting one factor of 7 from each of the multiples of 7 (these are $7, 14, 21, \ldots, 182, 189, 196$), we see that 200! includes at least 28 factors of 7.
However, each multiple of $7^2 = 49$ includes a second factor of 7 (since $49 = 7^2, 98 = 7^2 \times 2$, etc.) which was not counted in the previous 28 factors.
The product 200! includes 4 multiples of 49, since $4 \times 49 = 196$, and thus there are at least 4 additional factors of 7 in 200!.
Since $7^3 > 200$, then 200! does not include any multiples of 7^3 and so we have counted all possible factors of 7.
Thus, 200! includes exactly $28 + 4 = 32$ factors of 7, and so $200! = 7^{32} \times t$ for some positive

integer t that is not divisible by 7.
Counting in a similar way, the product 90! includes 12 multiples of 7 and 1 multiple of 49, and thus includes 13 factors of 7.
Therefore, $90! = 7^{13} \times r$ for some positive integer r that is not divisible by 7.
Also, 30! includes 4 factors of 7, and thus $30! = 7^4 \times s$ for some positive integer s that is not divisible by 7.
Therefore, $\dfrac{200!}{90!30!} = \dfrac{7^{32} \times t}{(7^{13} \times r)(7^4 \times s)} = \dfrac{7^{32} \times t}{(7^{17} \times rs)} = \dfrac{7^{15} \times t}{rs}$.
Since we are given that $\dfrac{200!}{90!30!}$ is equal to a positive integer, then $\dfrac{7^{15} \times t}{rs}$ is a positive integer.
Since r and s contain no factors of 7 and $7^{15} \times t$ is divisible by rs, then it must be the case that t is divisible by rs.
In other words, we can re-write $\dfrac{200!}{90!30!} = \dfrac{7^{15} \times t}{rs}$ as $\dfrac{200!}{90!30!} = 7^{15} \times \dfrac{t}{rs}$ where $\dfrac{t}{rs}$ is an integer.
Since each of r, s and t does not include any factors of 7, then the integer $\dfrac{t}{rs}$ is not divisible by 7.
Therefore, the largest power of 7 which divides $\dfrac{200!}{90!30!}$ is 7^{15}, and so $n = 15$.

ANSWER: 15

17. Let $BD = h$.
Since $\angle BCA = 45^\circ$ and $\triangle BDC$ is right-angled at D, then $\angle CBD = 180^\circ - 90^\circ - 45^\circ = 45^\circ$.
This means that $\triangle BDC$ is isosceles with $CD = BD = h$.
Since $\angle BAC = 60^\circ$ and $\triangle BAD$ is right-angled at D, then $\triangle BAD$ is a 30°-60°-90° triangle.
Therefore, $BD : DA = \sqrt{3} : 1$.
Since $BD = h$, then $DA = \dfrac{h}{\sqrt{3}}$.
Thus, $CA = CD + DA = h + \dfrac{h}{\sqrt{3}} = h\left(1 + \dfrac{1}{\sqrt{3}}\right) = \dfrac{h(\sqrt{3}+1)}{\sqrt{3}}$.
We are told that the area of $\triangle ABC$ is $72 + 72\sqrt{3}$.
Since BD is perpendicular to CA, then the area of $\triangle ABC$ equals $\dfrac{1}{2} \cdot CA \cdot BD$.
Thus,

$$\begin{aligned} \frac{1}{2} \cdot CA \cdot BD &= 72 + 72\sqrt{3} \\ \frac{1}{2} \cdot \frac{h(\sqrt{3}+1)}{\sqrt{3}} \cdot h &= 72(1+\sqrt{3}) \\ \frac{h^2}{2\sqrt{3}} &= 72 \\ h^2 &= 144\sqrt{3} \end{aligned}$$

and so $BD = h = 12\sqrt[4]{3}$ since $h > 0$.

ANSWER: $12\sqrt[4]{3}$

18. Since each word is to be 7 letters long and there are two choices for each letter, there are $2^7 = 128$ such words.
We count the number of words that do contain three or more A's in a row and subtract this total from 128.

There is 1 word with exactly 7 A's in a row: AAAAAAA.

There are 2 words with exactly 6 A's in a row: AAAAAAB and BAAAAAA.

Consider the words with exactly 5 A's in a row.
If such a word begins with exactly 5 A's, then the 6th letter is B and so the word has the form AAAAABx where x is either A or B. There are 2 such words.
If such a word has the string of exactly 5 A's beginning in the second position, then it must be BAAAAAB since there cannot be an A either immediately before or immediately after the 5 A's.
If such a word ends with exactly 5 A's, then the 2nd letter is B and so the word has the form xBAAAAA where x is either A or B. There are 2 such words.
There are 5 words wth exactly 5 A's in a row.

Consider the words with exactly 4 A's in a row.
Using similar reasoning, such a word can be of one of the following forms: AAAABxx, BAAAABx, xBAAAAB, xxBAAAA.
Since there are two choices for each x, then there are $4 + 2 + 2 + 4 = 12$ words.

Consider the words with exactly 3 A's in a row.
Using similar reasoning, such a word can be of one of the following forms: AAABxxx, BAAABxx, xBAAABx, xxBAAAB, xxxBAAA.
Since there are two choices for each x, then there appear to be $8 + 4 + 4 + 4 + 8 = 28$ such words. However, the word AAABAAA is counted twice. (This the only word counted twice in any of these cases.) Therefore, there are 27 such words.

In total, there are $1 + 2 + 5 + 12 + 27 = 47$ words that contain three or more A's in a row, and so there are $128 - 47 = 81$ words that do not contain three or more A's in a row.

ANSWER: 81

19. Let $t = x^{1/5}$. Thus, $x^{2/5} = t^2$ and $x^{3/5} = t^3$.
Therefore, the following equations are equivalent:

$$\begin{aligned} x^{3/5} - 4 &= 32 - x^{2/5} \\ t^3 + t^2 - 36 &= 0 \\ (t-3)(t^2 + 4t + 12) &= 0 \end{aligned}$$

Thus, $t = 3$ or $t^2 + 4t + 12 = 0$.
The discriminant of this quadratic equation is $4^2 - 4(1)(12) < 0$, which means that there are no real values of t that are solutions. This in turn means that there are no corresponding real values of x.
This gives $x^{1/5} = t = 3$ and so $x = 3^5 = 243$ is the only solution.

ANSWER: $3^5 = 243$

20. Let $AC = x$.
Thus $BC = AC - 1 = x - 1$.
Since $AC = AB - 1$, then $AB = AC + 1 = x + 1$.
The perimeter of $\triangle ABC$ is $BC + AC + AB = (x - 1) + x + (x + 1) = 3x$.
By the cosine law in $\triangle ABC$,

$$\begin{aligned} BC^2 &= AB^2 + AC^2 - 2(AB)(AC)\cos(\angle BAC) \\ (x-1)^2 &= (x+1)^2 + x^2 - 2(x+1)x\left(\tfrac{3}{5}\right) \\ x^2 - 2x + 1 &= x^2 + 2x + 1 + x^2 - \tfrac{6}{5}(x^2 + x) \\ 0 &= x^2 + 4x - \tfrac{6}{5}(x^2 + x) \\ 0 &= 5x^2 + 20x - 6(x^2 + x) \\ x^2 &= 14x \end{aligned}$$

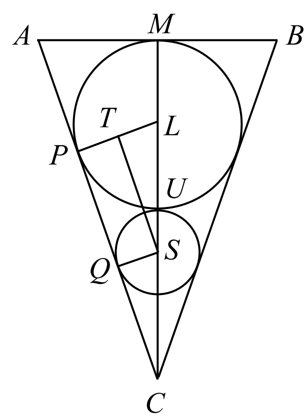
Since $x > 0$, then $x = 14$.
Therefore, the perimeter of $\triangle ABC$, which equals $3x$, is 42.

Answer: 42

21. Since $f(2x-3) - 2f(3x-10) + f(x-3) = 28 - 6x$ for all real numbers x, then when $x = 2$, we obtain $f(2(2)-3) - 2f(3(2)-10) + f(2-3) = 28 - 6(2)$ and so $f(1) - 2f(-4) + f(-1) = 16$.
Since f is an odd function, then $f(-1) = -f(1)$ or $f(1) + f(-1) = 0$.
Combining with $f(1) - 2f(-4) + f(-1) = 16$, we obtain $-2f(-4) = 16$ and so $-f(-4) = 8$.
Since f is an odd function, then $f(4) = -f(-4) = 8$.

Answer: 8

22. Let the radius of the small sphere be r and the radius of the large sphere be $2r$.
Draw a vertical cross-section through the centre of the top face of the cone and its bottom vertex.
By symmetry, this will pass through the centres of the spheres.
In the cross-section, the cone becomes a triangle and the spheres become circles.



We label the vertices of the triangle A, B, C.
We label the centres of the large circle and small circle L and S, respectively.
We label the point where the circles touch U.
We label the midpoint of AB (which represents the centre of the top face of the cone) as M.
Join L and S to the points of tangency of the circles to AC. We call these points P and Q.
Since LP and SQ are radii, then they are perpendicular to the tangent line AC at P and Q,

respectively.
Draw a perpendicular from S to T on LP.
The volume of the cone equals $\frac{1}{3}\pi \cdot AM^2 \cdot MC$. We determine the lengths of AM and MC in terms of r.
Since the radii of the small circle is r, then $QS = US = r$.
Since $TPQS$ has three right angles (at T, P and Q), then it has four right angles, and so is a rectangle.
Therefore, $PT = QS = r$.
Since the radius of the large circle is $2r$, then $PL = UL = ML = 2r$.
Therefore, $TL = PL - PT = 2r - r = r$.
Since MC passes through L and S, it also passes through U, the point of tangency of the two circles.
Therefore, $LS = UL + US = 2r + r = 3r$.
By the Pythagorean Theorem in $\triangle LTS$,

$$TS = \sqrt{LS^2 - TL^2} = \sqrt{(3r)^2 - r^2} = \sqrt{8r^2} = 2\sqrt{2}r$$

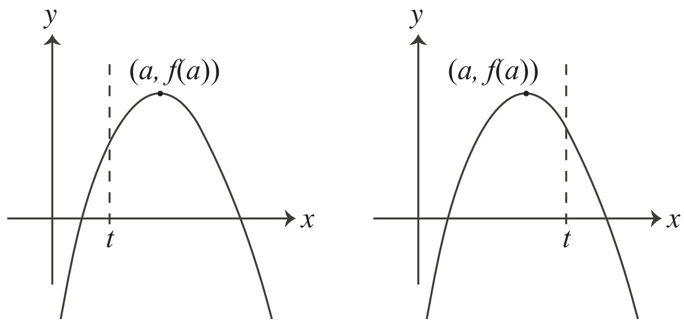
since $TS > 0$ and $r > 0$.
Consider $\triangle LTS$ and $\triangle SQC$.
Each is right-angled, $\angle TLS = \angle QSC$ (because LP and QS are parallel), and $TL = QS$.
Therefore, $\triangle LTS$ is congruent to $\triangle SQC$.
Thus, $SC = LS = 3r$ and $QC = TS = 2\sqrt{2}r$.
This tells us that $MC = ML + LS + SC = 2r + 3r + 3r = 8r$.
Also, $\triangle AMC$ is similar to $\triangle SQC$, since each is right-angled and they have a common angle at C.
Therefore, $\dfrac{AM}{MC} = \dfrac{QS}{QC}$ and so $AM = \dfrac{8r \cdot r}{2\sqrt{2}r} = 2\sqrt{2}r$.
This means that the volume of the original cone is $\frac{1}{3}\pi \cdot AM^2 \cdot MC = \frac{1}{3}\pi(2\sqrt{2}r)^2(8r) = \frac{64}{3}\pi r^3$.
The volume of the large sphere is $\frac{4}{3}\pi(2r)^3 = \frac{32}{3}\pi r^3$.
The volume of the small sphere is $\frac{4}{3}\pi r^3$.
The volume of the cone not occupied by the spheres is $\frac{64}{3}\pi r^3 - \frac{32}{3}\pi r^3 - \frac{4}{3}\pi r^3 = \frac{28}{3}\pi r^3$.
The fraction of the volume of the cone that this represents is $\dfrac{\frac{28}{3}\pi r^3}{\frac{64}{3}\pi r^3} = \frac{28}{64} = \frac{7}{16}$.

Answer: $\frac{7}{16}$

23. Let $f(x) = -x^2 + 2ax + a$.
Since $(x-a)^2 = x^2 - 2ax + a^2$, then

$$-x^2 + 2ax + a = -(x^2 - 2ax + a^2) + a^2 + a = -(x-a)^2 + a^2 + a$$

Therefore, $M(t)$ is the maximum value of $-(x-a)^2 + a^2 + a$ over all real numbers x with $x \leq t$.
Now the graph of $y = f(x) = -(x-a)^2 + a^2 + a$ is a parabola opening downwards with vertex at $(a, a^2 + a)$.
Since the parabola opens downwards, then the parabola reaches its maximum at the vertex $(a, a^2 + a)$.
Therefore, $f(x)$ is increasing when $x < a$ and decreasing when $x > a$.
This means that, when $t < a$, the maximum of the values of $f(x)$ with $x \leq t$ is $f(t)$ (because $f(x)$ increases until $f(t)$) and when $t \geq a$, the maximum of the values of $f(x)$ with $x \leq t$ is $f(a)$ (because the maximum value of $f(x)$ is to the left of t).



In other words,

$$M(t) = \begin{cases} f(t) & t < a \\ f(a) & t \geq a \end{cases}$$

Since $a - 1 < a$, then $M(a-1) = f(a-1)$.
Since $a + 2 > a$, then $M(a+2) = f(a)$.
Therefore,

$$\begin{aligned} M(a-1) + M(a+2) &= f(a-1) + f(a) \\ &= (-((a-1) - a)^2 + a^2 + a) + (-(a-a)^2 + a^2 + a) \\ &= -1 + a^2 + a - 0 + a^2 + a \\ &= 2a^2 + 2a - 1 \end{aligned}$$

ANSWER: $2a^2 + 2a - 1$

24. We find the points of intersection of $y = 2\cos 3x + 1$ and $y = -\cos 2x$ by equating values of y and obtaining the following equivalent equations:

$$\begin{aligned}
2\cos 3x + 1 &= -\cos 2x \\
2\cos(2x + x) + \cos 2x + 1 &= 0 \\
2(\cos 2x \cos x - \sin 2x \sin x) + \cos 2x + 1 &= 0 \\
2((2\cos^2 x - 1)\cos x - (2\sin x \cos x)\sin x) + (2\cos^2 x - 1) + 1 &= 0 \\
2(2\cos^3 x - \cos x - 2\sin^2 x \cos x) + 2\cos^2 x &= 0 \\
4\cos^3 x - 2\cos x - 4(1 - \cos^2 x)\cos x + 2\cos^2 x &= 0 \\
4\cos^3 x - 2\cos x - 4\cos x + 4\cos^3 x + 2\cos^2 x &= 0 \\
8\cos^3 x + 2\cos^2 x - 6\cos x &= 0 \\
4\cos^3 x + \cos^2 x - 3\cos x &= 0 \\
\cos x(4\cos^2 x + \cos x - 3) &= 0 \\
\cos x(\cos x + 1)(4\cos x - 3) &= 0
\end{aligned}$$

Therefore, $\cos x = 0$ or $\cos x = -1$ or $\cos x = \frac{3}{4}$.
Two of these points of intersection, P and Q, have x-coordinates between $\frac{17\pi}{4} = 4\pi + \frac{\pi}{4}$ and $\frac{21\pi}{4} = 5\pi + \frac{\pi}{4}$.
Since $\cos 4\pi = 1$ and $\cos(4\pi + \frac{\pi}{4}) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{2\sqrt{2}}{4} < \frac{3}{4}$, then there is an angle θ between 4π and $\frac{17\pi}{4}$ with $\cos\theta = \frac{3}{4}$ and so there is not such an angle between $\frac{17\pi}{4}$ and $\frac{21\pi}{4}$.
Therefore, we look for angles x between $\frac{17\pi}{4}$ and $\frac{21\pi}{4}$ with $\cos x = 0$ and $\cos x = -1$.
Note that $\cos 5\pi = -1$ and that $\cos\frac{18\pi}{4} = \cos\frac{9\pi}{2} = \cos\frac{\pi}{2} = 0$.
Thus, the x-coordinates of P and Q are 5π and $\frac{9\pi}{2}$.
Suppose that P has x-coordinate 5π.
Since P lies on $y = -\cos 2x$, then its y-coordinate is $y = -\cos 10\pi = -1$.
Suppose that Q has x-coordinate $\frac{9\pi}{2}$.
Since Q lies on $y = -\cos 2x$, then its y-coordinate is $y = -\cos 9\pi = 1$.
Therefore, the coordinates of P are $(5\pi, -1)$ and the coordinates of Q are $(\frac{9\pi}{2}, 1)$.
The slope of the line through P and Q is $\dfrac{1 - (-1)}{\frac{9\pi}{2} - 5\pi} = \dfrac{2}{-\frac{\pi}{2}} = -\frac{4}{\pi}$.
The line passes through the point with coordinates $(5\pi, -1)$.
Thus, its equation is $y - (-1) = -\frac{4}{\pi}(x - 5\pi)$ or $y + 1 = -\frac{4}{\pi}x + 20$ or $y = -\frac{4}{\pi}x + 19$.
This line intersects the y-axis at $A(0, 19)$. Thus, $AO = 19$.
This line intersects the x-axis at point B with y-coordinate 0 and hence the x-coordinate of B is $x = 19 \cdot \frac{\pi}{4} = \frac{19\pi}{4}$. Thus, $BO = \frac{19\pi}{4}$.
Since $\triangle AOB$ is right-angled at the origin, then its area equals $\frac{1}{2}(AO)(BO) = \frac{1}{2}(19)(\frac{19\pi}{4}) = \frac{361\pi}{8}$.

ANSWER: $\frac{361\pi}{8}$

25. Let $f(2n)$ be the number of different ways to draw n non-intersecting line segments connecting pairs of points so that each of the $2n$ points is connected to exactly one other point.
Then $f(2) = 1$ (since there are only 2 points) and $f(6) = 5$ (from the given example).
Also, $f(4) = 2$. (Can you see why?)
We show that

$$f(2n) = f(2n-2)+f(2)f(2n-4)+f(4)f(2n-6)+f(6)f(2n-8)+\cdots+f(2n-4)f(2)+f(2n-2)$$

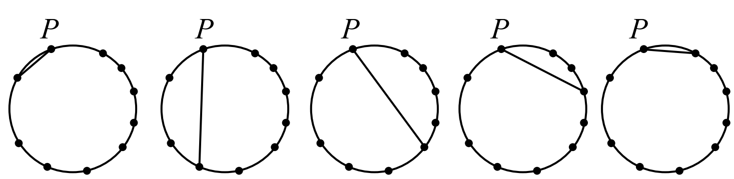
Here is a justification for this equation:

Pick one of the $2n$ points and call it P.
P could be connected to the 1st point counter-clockwise from P. This leaves $2n-2$ points on the circle. By definition, these can be connected in $f(2n-2)$ ways.
P cannot be connected to the 2nd point counter-clockwise, because this would leave an odd number of points on one side of this line segment. An odd number of points cannot be connected in pairs as required.
Similarly, P cannot be connected to any of the 4th, 6th, 8th, $(2n-2)$th points.
P can be connected to the 3rd point counter-clockwise, leaving 2 points on one side and $2n-4$ points on the other side. There are $f(2)$ ways to connect the 2 points and $f(2n-4)$ ways to connect the $2n-4$ points. Therefore, in this case there are $f(2)f(2n-4)$ ways to connect the points. We cannot connect a point on one side of the line to a point on the other side of the line because the line segments would cross, which is not allowed.
P can be connected to the 5th point counter-clockwise, leaving 4 points on one side and $2n-6$ points on the other side. There are $f(4)$ ways to connect the 4 points and $f(2n-6)$ ways to connect the $2n-6$ points. Therefore, in this case there are $f(4)f(2n-6)$ ways to connect the points.
Continuing in this way, P can be connected to every other point until we reach the last ($(2n-1)$th) point which will leave $2n-2$ points on one side and none on the other. There are $f(2n-2)$ possibilities in this case.
Adding up all of the cases, we see that there are

$$f(2n) = f(2n-2)+f(2)f(2n-4)+f(4)f(2n-6)+f(6)f(2n-8)+\cdots+f(2n-4)f(2)+f(2n-2)$$

ways of connecting the points.

The figures below show the case of $2n = 10$.



We can use this formula to successively calculate $f(8), f(10), f(12), f(14), f(16)$:

$$\begin{aligned} f(8) &= f(6) + f(2)f(4) + f(4)f(2) + f(6) \\ &= 5 + 1(2) + 2(1) + 5 \\ &= 14 \end{aligned}$$

$$\begin{aligned} f(10) &= f(8) + f(2)f(6) + f(4)f(4) + f(6)f(2) + f(8) \\ &= 14 + 1(5) + 2(2) + 5(1) + 14 \\ &= 42 \end{aligned}$$

$$\begin{aligned} f(12) &= f(10) + f(2)f(8) + f(4)f(6) + f(6)f(4) + f(8)f(2) + f(10) \\ &= 42 + 1(14) + 2(5) + 5(2) + 14(1) + 42 \\ &= 132 \end{aligned}$$

$$\begin{aligned} f(14) &= f(12) + f(2)f(10) + f(4)f(8) + f(6)f(6) + f(8)f(4) + f(10)f(2) + f(12) \\ &= 132 + 1(42) + 2(14) + 5(5) + 14(2) + 42(1) + 132 \\ &= 429 \end{aligned}$$

$$\begin{aligned} f(16) &= f(14) + f(2)f(12) + f(4)f(10) + f(6)f(8) + f(8)f(6) + f(10)f(4) + f(12)f(2) + f(14) \\ &= 429 + 1(132) + 2(42) + 5(14) + 14(5) + 42(2) + 132(1) + 429 \\ &= 1430 \end{aligned}$$

Therefore, there are 1430 ways to join the 16 points.
(The sequence $1, 2, 5, 12, 42, \ldots$ is a famous sequence called the *Catalan numbers*.)

ANSWER: 1430

Relay Problems

(Note: Where possible, the solutions to parts (b) and (c) of each Relay are written as if the value of t is not initially known, and then t is substituted at the end.)

0. (a) Evaluating, $\dfrac{9+3\times 3}{3} = \dfrac{9+9}{3} = \dfrac{18}{3} = 6$.

   (b) The area of a triangle with base $2t$ and height $3t+2$ is $\frac{1}{2}(2t)(3t+2)$ or $t(3t+2)$.
   Since the answer to (a) is 6, then $t=6$, and so $t(3t+2) = 6(20) = 120$.

   (c) Since $AB = BC$, then $\angle BCA = \angle BAC$.
   Since $\angle ABC = t^\circ$, then $\angle BAC = \frac{1}{2}(180^\circ - \angle ABC) = \frac{1}{2}(180^\circ - t^\circ) = 90^\circ - \frac{1}{2}t^\circ$.
   Since the answer to (b) is 120, then $t = 120$, and so

$$\angle BAC = 90^\circ - \tfrac{1}{2}(120^\circ) = 30^\circ$$

Answer: $6, 120, 30^\circ$

1. (a) We find the prime factorization of 390:

$$390 = 39\cdot 10 = 3\cdot 13\cdot 2\cdot 5$$

   Since 9450 is divisible by 10, then 2 and 5 are also prime factors of 9450.
   Since the sum of the digits of 9450 is $9+4+5+0=18$ which is a multiple of 3, then 9450 is also divisible by 3.
   (We can check that 9450 is not divisible by 13.)
   Therefore, the sum of the three common prime divisors of 390 and 9450 is $2+3+5=10$.

   (b) Simplifying,

$$\begin{aligned} n &= \frac{(4t^2-10t-2)-3(t^2-t+3)+(t^2+5t-1)}{(t+7)+(t-13)} \\ &= \frac{4t^2-10t-2-3t^2+3t-9+t^2+5t-1}{2t-6} \\ &= \frac{2t^2-2t-12}{2t-6} \\ &= \frac{t^2-t-6}{t-3} \\ &= \frac{(t-3)(t+2)}{t-3} \\ &= t+2 \end{aligned}$$

   assuming that $t\neq 3$.
   Since the answer to (a) is 10, then $t=10$, and so $n = t+2 = 12$.

   (c) We determine the average by calculating the sum of the 36 possible sums, and then dividing by 36.
   To determine the sum of the 36 possible sums, we determine the sum of the 36 values that appear on the top face of each of the two dice.
   Each of the 6 faces on the first dice is rolled in 6 of the 36 possibilities.
   Thus, these faces contribute $6(1+2+3+4+5+6) = 6(21) = 126$ to the sum of the 36 possible sums.
   Each of the 6 sides on the second dice is rolled in 6 of the 36 possibilities.

Thus, these faces contribute $6((t-10)+t+(t+10)+(t+20)+(t+30)+(t+40))$ or $6(6t+90)$ or $36t+540$ to the sum of the 36 possible sums.
Therefore, the sum of the 36 sums is $126+36t+540=36t+666$, and so the average of the 36 sums is $\frac{36t+666}{36}=t+\frac{111}{6}=t+\frac{37}{2}$.
Since the answer to (b) is 12, then $t=12$ and so the average of the 36 possible sums is $12+\frac{37}{2}=\frac{61}{2}=30.5$.

ANSWER: $10, 12, \frac{61}{2}$

2. (a) Expanding and simplifying,

$$2(x-3)^2-12=2(x^2-6x+9)-12=2x^2-12x+6$$

Thus, $a=2$, $b=-12$, and $c=6$.
This means that $10a-b-4c=10(2)-(-12)-4(6)=8$.

(b) The line through the points $(11,-7)$ and $(15,5)$ has slope $\frac{5-(-7)}{15-11}=\frac{12}{4}=3$.
Thus, a line perpendicular to this line has slope $-\frac{1}{3}$.
Therefore, the slope of the line through the points $(-4,t)$ and (k,k) has slope $-\frac{1}{3}$.
Thus, $\frac{k-t}{k-(-4)}=-\frac{1}{3}$.
We solve for k:

$$\begin{aligned}\frac{k-t}{k+4}&=-\frac{1}{3}\\3k-3t&=-k-4\\4k&=3t-4\\k&=\frac{3}{4}t-1\end{aligned}$$

Since the answer to (a) is 8, then $t=8$ and so $k=\frac{3}{4}(8)-1=5$.

(c) The sum of the entries in the second row is

$$(4t-1)+(2t+12)+(t+16)+(3t+1)=10t+28$$

This means that the sum of the four entries in any row, column or diagonal will also be $10t+28$.
Looking at the fourth column, the top right entry is thus

$$10t+28-(3t+1)-(t+15)-(4t-5)=2t+17$$

Looking at the top row, the top left entry is thus

$$10t+28-(3t-2)-(4t-6)-(2t+17)=t+19$$

Looking at the southeast diagonal, the third entry is thus

$$10t+28-(t+19)-(2t+12)-(4t-5)=3t+2$$

Looking at the third row,

$$N = 10t + 28 - (4t - 2) - (3t + 2) - (t + 15) = 2t + 13$$

Since the answer to (b) is 5, then $t = 5$. Thus, $N = 23$.
We note that we can complete the grid, both in terms of t and using $t = 5$ as follows:

$t+19$	$3t-2$	$4t-6$	$2t+17$
$4t-1$	$2t+12$	$t+16$	$3t+1$
$2t+13$	$4t-2$	$3t+2$	$t+15$
$3t-3$	$t+20$	$2t+16$	$4t-5$

24	13	14	27
19	22	21	16
23	18	17	20
12	25	26	15

ANSWER: $8, 5, 23$

3. (a) Since (a, a^2) lies on the line with equation $y = 5x + a$, then $a^2 = 5a + a$ or $a^2 = 6a$.
Since $a \neq 0$, then $a = 6$.

(b) The team scored a total of $10t \cdot 4 + 20 \cdot g = 40t + 20g$ points over their $4 + g$ games.
Since their average number of points per game was 28, then

$$\begin{aligned}\frac{40t + 20g}{g + 4} &= 28 \\ 40t + 20g &= 28g + 112 \\ 40t - 112 &= 8g \\ g &= 5t - 14\end{aligned}$$

Since the answer to (a) is 6, then $t = 6$ and so $g = 5(6) - 14 = 16$.

(c) Since $(x, y) = (a, b)$ satisfies the system of equations, then

$$\begin{aligned}a^2 + 4b &= t^2 \\ a^2 - b^2 &= 4\end{aligned}$$

Subtracting the second equation from the first, we obtain successively

$$\begin{aligned}(a^2 + 4b) - (a^2 - b^2) &= t^2 - 4 \\ b^2 + 4b &= t^2 - 4 \\ b^2 + 4b + 4 &= t^2 \\ (b + 2)^2 &= t^2 \\ b + 2 &= \pm t \\ b &= -2 \pm t\end{aligned}$$

Since the answer to (b) is 16, then $t = 16$.
Therefore, $b = -2 + t = 14$ or $b = -2 - t = -18$.
Since $b > 0$, then $b = 14$.

ANSWER: $6, 16, 14$

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Answer Key for Team Problems

Question	Answer
1	6
2	13.5 L
3	53
4	792
5	40°
6	639
7	2018 (accept $2^{11} - 2^5 + 2$)
8	13
9	$\frac{4}{3}$
10	526
11	20%
12	0
13	10
14	1200
15	6
16	15
17	$12\sqrt[4]{3}$
18	81
19	243 (accept 3^5)
20	42
21	8
22	$\frac{7}{16}$
23	$2a^2 + 2a - 1$
24	$\frac{361\pi}{8}$
25	1430

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2018 Canadian Team Mathematics Contest

Answer Key for Individual Problems

Question	Answer
1	-8
2	$\frac{1}{12}$ (accept 0.083 or more precise)
3	3
4	50
5	870
6	$(3, 6), (39, -78)$
7	191
8	$18\sqrt{5}$ m (units not required; accept 40.2 or more precise)
9	$(105, 91)$
10	$(1250, 1500, 750, 3)$

Answer Key for Relays

Question	Answer
0	$6, 120, 30^\circ$
1	$10, 12, 30.5$
2	$8, 5, 23$
3	$6, 16, 14$